Darcy's Law: \[ \vec{q}_v = -K \nabla h \]

HYDRAULIC CONDUCTIVITY, \( K \) m/s

\[ K = k \rho g / \mu \]

= kg/ν units of velocity

Proportionality constant in Darcy's Law

Property of both fluid and medium

see D&S, p. 62
HYDRAULIC POTENTIAL (Φ): energy/unit mass

\[ \Phi = g h = g z + P/\rho_w \]

Consider incompressible fluid element

@ elevation \( z_i = 0 \) pressure \( P_i \), \( \rho_i \) and velocity \( v = 0 \)

Move to new position \( z, P, \rho, v \)

Energy difference: lift mass + accelerate + compress (= \( \int VdP \))

\[ = mg(z - z_i) + \frac{mv^2}{2} + m \int \frac{V}{m} dP \]
latter term = \( m \int (1/\rho)dP \)

Energy/unit mass \( \Phi = g z + \frac{v^2}{2} + \int (1/\rho)dP \)

For incompressible fluid (\( \rho = \text{const} \)) & slow flow (\( v^2/2 \rightarrow 0 \)), \( z_i=0, \ P_i = 0 \)

Energy/unit mass: \( \Phi = g z + \frac{P}{\rho} = g h \)

Force/unit mass = \( \nabla \Phi = g - \nabla P/\rho \)

Force/unit weight = \( \nabla h = 1 - \nabla P/\rho g \)
Rewrite Darcy's Law: 

\[ \vec{q}_m = \rho \vec{q}_v = \frac{k \rho}{\nu} \left[ g - \frac{\nabla P}{\rho} \right] \]

\[ = \frac{k \rho}{\nu} \text{[force/unit mass]} \]

\[ \vec{q}_m \equiv \text{Fluid flux mass vector (g/cm}^2\text{-sec)} \]
\[ \propto k \equiv \text{rock (matrix) permeability (cm}^2\text{)} \]
\[ \propto \rho \equiv \text{fluid density (g/cm}^3\text{)} \]
\[ \propto [.....] \equiv \text{Force/unit mass acting on fluid element} \]
\[ \propto 1/\nu \]

where \( \nu \equiv \text{Kinematic Viscosity} \)
\[ = \frac{\mu}{\rho} \text{ cm}^2/\text{sec} \]
Rewrite Darcy's Law: Hubbert (1940; J. Geol. 48, p. 785-944)

\[ \vec{q}_v = \frac{k}{\rho \nu} [\rho g - \nabla P] \]

\[ = \frac{k}{\rho \nu} [\text{force/unit vol}] \]

\[ \vec{q}_v \equiv \text{Fluid volumetric flux vector (cm}^3/\text{cm}^2\text{-sec)} = q_m/\rho \]
\[ \propto \text{ k } \equiv \text{rock (matrix) permeability (cm}^2\) \]
\[ \propto [.....] \equiv \text{Force/unit vol. acting on fluid element} \]
\[ \propto 1/\nu \]

where \( \nu \equiv \text{Kinematic Viscosity} \]
\[ = \frac{\mu}{\rho} \text{ cm}^2/\text{sec} \]
Rewrite Darcy's Law: 

\[ \vec{q}_v = \frac{k}{\rho v} [\rho g - \nabla P] \]

\[ = \frac{k}{\rho v} [\rho g \nabla h] \]

\[ = \frac{kg}{v} \nabla h \]

\[ = K \nabla h \]
STATIC FLUID (NO FLOW)

\[ \vec{q}_m = \frac{k\rho}{\nu} \left[ \frac{g - \nabla P}{\rho} \right] \]
STATIC FLUID (NO FLOW)

\[ q_m = \frac{k \rho}{\nu} \left[ g - \frac{\nabla P}{\rho} \right] \]

Force/unit mass = 0 for \( q_m = 0 \)

\[ \frac{\partial P}{\partial z} = \rho g \quad \frac{\partial P}{\partial x} = 0 \quad \frac{\partial P}{\partial y} = 0 \]

Converse:
Horizontal pressure gradients require fluid flow
Darcy's Law: Isotropic Media: \( \vec{q} = -K \nabla h \)
OK only if \( K_x = K_y = K_z \)

Darcy's Law: Anisotropic Media
\( K, k \) are tensors

Direction of fluid flow need not coincide with the gradient in hydraulic head
Darcy's Law: Isotropic Media: \( \mathbf{q} = - \mathbf{K} \nabla h \)

OK only if \( K_x = K_y = K_z \)

Darcy's Law: Anisotropic Media

\( \mathbf{K} \) is a tensor

Simplest case (orthorhombic?)

where principal directions of anisotropy coincide with \( x, y, z \)

\[
\mathbf{q} = - \begin{bmatrix} K_{xx} & 0 & 0 \\ 0 & K_{yy} & 0 \\ 0 & 0 & K_{zz} \end{bmatrix} \begin{pmatrix} \hat{i} \frac{\partial h}{\partial x} \\ \hat{j} \frac{\partial h}{\partial y} \\ \hat{k} \frac{\partial h}{\partial z} \end{pmatrix}
\]

Thus

\[
\mathbf{q}_x = - K_{xx} \frac{\partial h}{\partial x} \hat{i} \quad \mathbf{q}_y = - K_{yy} \frac{\partial h}{\partial y} \hat{j} \quad \mathbf{q}_z = - K_{zz} \frac{\partial h}{\partial z} \hat{k}
\]
General case: Symmetrical tensor

\[ K_{xy} = K_{yx} \quad K_{zx} = K_{xz} \quad K_{yz} = K_{zy} \]

\[ \mathbf{q} = - \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix} \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial z} \end{pmatrix} \]

\[ q_x = - K_{xx} \frac{\partial h}{\partial x} - K_{xy} \frac{\partial h}{\partial y} - K_{xz} \frac{\partial h}{\partial z} \]

\[ q_y = - K_{yx} \frac{\partial h}{\partial x} - K_{yy} \frac{\partial h}{\partial y} - K_{yz} \frac{\partial h}{\partial z} \]

\[ q_z = - K_{zx} \frac{\partial h}{\partial x} - K_{zy} \frac{\partial h}{\partial y} - K_{zz} \frac{\partial h}{\partial z} \]
# Relevant Physical Properties for Darcy’s Law

<table>
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</table>

## “Darcy” Version

$$
\vec{q}_v = - K \nabla h
$$

## Hubbert Version

$$
\vec{q}_v = \frac{k}{\rho \nu} \left[ \rho g - \nabla P \right]
$$

$$
\vec{q}_m = \rho \vec{q}_v
$$
Relevant Physical Properties for Darcy’s Law

**Hydraulic conductivity (K)** \( \text{cm/s} \)

Units of velocity
Proportionality constant in Darcy’s Law
Property of both fluid and medium

\[ \vec{q}_v = -K \nabla h \]

\[ \vec{q}_v = \frac{kg}{\nu} \left[ 1 - \frac{\nabla P}{\rho g} \right] \]

\[ \vec{q}_m = \rho \vec{q}_v \]

\( \Rightarrow \) \( K = \frac{\text{kg}}{\text{\nu}} \)

and where \( \nabla h = 1 - \nabla P/\rho g \)
DENSITY \((\rho) \quad \text{g/cm}^3\)

Fluid property

Specific weight (weight density) \(\gamma = \rho \ g\)

\(\rho = f(T,P)\)

\[
\rho_{T,P} \equiv \rho_o \left\{1 - \alpha (T-T_o) + \beta (P-P_o)\right\} \text{ for small } \alpha, \beta
\]

where

Thermal expansivity

\[
\alpha \equiv \frac{1}{V} \left(\frac{\partial V}{\partial T}\right)_P = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T}\right)_P \quad \text{because} \quad \frac{d\rho}{\rho} = -\frac{dV}{V}
\]

Isothermal Compressibility

\[
\beta_T \equiv -\frac{1}{V} \left(\frac{\partial V}{\partial P}\right)_T = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial P}\right)_T
\]
DYNAMIC VISCOSITY $\mu$  Fluid property

Geo. Stokes Law:

$$\frac{4\pi r^3}{3} (\rho_s - \rho_f) g = 6\pi r \mu u$$

Gravitational Force = Frictional Force

Units of $\mu$:

- poise; 1 P = 0.1 N sec/m$^2$
- $= 1$ dyne sec/cm$^2$

Water 0.01 poise (1 centipoise)
DYNAMIC VISCOSITY  \( \mu \)  
Fluid property

A measure of the rate of strain in an imperfectly elastic material subjected to a distortional stress.

For simple shear  \( \tau = \mu \partial u/\partial y \)  
Newtonian fluid

Units  (poise; 1 P = 0.1 N sec/m\(^2\) = 1 dyne sec/cm\(^2\)) 
Water  0.01 poise (1 centipoise)

KINEMATIC VISCOSITY  \( \nu \)  
Fluid property

\[ \nu = \frac{\mu}{\rho} \text{ m}^2/\text{sec} \text{ or cm}^2/\text{sec} \]

Water:  \( 10^{-6} \text{ m}^2/\text{sec} = 10^{-2} \text{ cm}^2/\text{sec} \)
Basaltic Magma  0.1 m\(^2\)/sec
Asphalt @ 20°C  
or granitic magma  \( 10^2 \text{ m}^2/\text{sec} \)
Mantle  \( 10^{16} \text{ m}^2/\text{sec} \)  
see Tritton p. 5; Elder p. 221)
Darcy's Law: 

\[ \vec{q}_v = \frac{kg}{\nu} \left[ 1 - \frac{\nabla P}{g \rho} \right] = -\frac{kg}{\nu} [\nabla h] = -K \nabla h \]

where:

\( \vec{q}_v \equiv \text{Darcy Velocity, Specific Discharge} \)
\( \text{or Fluid volumetric flux vector (cm/sec)} \)

\( k \equiv \text{permeability (cm}^2\text{)} \)

\( K = k(g/\nu) \equiv \text{hydraulic conductivity (cm/sec)} \)

\( \nu \equiv \text{Kinematic viscosity, cm}^2/\text{sec} \)
POROSITY \( (\phi, \text{ or } n) \) \hspace{1cm} \text{dimensionless} \hspace{1cm} \text{Rock property}

Ratio of void space to total volume of material

\[ \phi = \frac{V_v}{V_T} \]

Dictates how much water a saturated material can contain

Large Range: \(<0.1\% \text{ to } >75\%\)
Strange behaviors

Important influence on bulk properties of material
\( \text{e.g., bulk r, heat capacity, seismic velocity} \ldots \)

Difference between Darcy velocity and average microscopic velocity

Decreases with depth: Shales \( \phi = \phi_o e^{-cz} \) exponential

Sandstones: \( \phi = \phi_o - cz \) linear
Porosity, %

Non-uniform grain sizes

FCC 26%

BCC 32%

Simple cubic 47.6%

Gravel
Sand
Silt & Clay

Shale Sandstone Siltstone

Limestone & Dolostone karstic

Fractured crystalline rocks Basalt

Pumice
PERMEABILITY \( (k) \) \( \text{cm}^2 \)

Measure of the ability of a material to transmit fluid under a hydrostatic gradient

Differences with Porosity?
PERMEABILITY $(k)$ $\text{cm}^2$

Measure of the ability of a material to transmit fluid under a hydrostatic gradient

Differences with Porosity?

Different Units

Styrofoam cup: High $\phi$, Low $k$

Uniform spheres: $\phi \neq f(\text{dia})$; $k \sim \text{dia}^2$
PERMEABILITY \( (k) \) cm\(^2\)

Measure of the ability of a material to transmit fluid under a hydrostatic gradient

Most important rock parameter pertinent to fluid flow

Relates to the presence of fractures and interconnected voids

\[
1 \text{ darcy} = 0.987 \times 10^{-8} \text{ cm}^2 = 0.987 \times 10^{-12} \text{ m}^2
\]

(e.g., sandstone)

Approximate relation between \( K \) and \( k \) (for cool water):

\[
K_{\text{m/s}} \approx 10^7 \, k \text{ m}^2 = 10^3 \, k \text{ cm}^2 = 10^{-5} \, k_{\text{darcy}}
\]

\[
K_{\text{cm/s}} \approx 10^5 \, k \text{ cm}^2 = 10^{-3} \, k_{\text{darcy}}
\]

\[
K_{\text{ft/y}} \approx 10^{11} \, k \text{ cm}^2
\]
Permeability, cm$^2$
GEOLOGIC REALITIES OF PERMEABILITY \( (k) \)

**Huge Range** in common geologic materials \( > 10^{13} \times \)

Decreases super-exponentially with depth

\[ k = C d^2 \] for granular material,  
where \( d = \) grain diameter, \( C \) is complicated parameter

\[ k = a^3/12L \] for parallel fractures of aperture width “a” and spacing \( L \)

\( k \) is *dynamic* (dissolution/precipitation, cementation,  
thermal or mechanical fracturing; plastic deformation)

\( K \) is very low in deforming rocks as cracks seal (marbles, halite)

Scale dependence:

\[ k_{\text{regional}} \geq k_{\text{most permeable parts of drill holes}} \gg k_{\text{lab; small scale}} \]